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A Proof of the Elliptic-Function Addition-Theorem.

BY J. C. FIELDS.

The integration of the differential equation

$$\frac{d\phi}{\Delta\phi} + \frac{d\psi}{\Delta\psi} = 0 \quad (\Delta\phi = \sqrt{1 - k^2 \sin^2 \phi}, \text{ etc.})$$

can be effected by aid of the factor

$$\frac{\Delta\phi\Delta\psi - k^2 \sin \phi \cos \phi \sin \psi \cos \psi}{1 - k^2 \sin^2 \phi \sin^2 \psi},$$

which, as will shortly be seen, is an integrating factor; thus,

$$\begin{aligned} 0 &= \frac{\Delta\phi\Delta\psi - k^2 \sin \phi \cos \phi \sin \psi \cos \psi}{1 - k^2 \sin^2 \phi \sin^2 \psi} \left(\frac{d\phi}{\Delta\phi} + \frac{d\psi}{\Delta\psi} \right) \\ &= \frac{1}{1 - k^2 \sin^2 \phi \sin^2 \psi} \left(\Delta\psi d\phi - k^2 \sin \phi \cos \phi \sin \psi \cos \psi \frac{d\psi}{\Delta\psi} \right) + (---);* \end{aligned}$$

and since $1 - k^2 \sin^2 \phi \sin^2 \psi = \cos^2 \phi + \sin^2 \phi - k^2 \sin^2 \phi \sin^2 \psi = \cos^2 \phi + \sin^2 \phi \Delta^2 \psi$,

$$\begin{aligned} \text{this} &= \frac{1}{\cos^2 \phi + \sin^2 \phi \Delta^2 \psi} \left(\Delta\psi d\phi - k^2 \sin \phi \cos \phi \sin \psi \cos \psi \frac{d\psi}{\Delta\psi} \right) + (---) \\ &= \frac{1}{1 + \tan^2 \phi \Delta^2 \psi} \left(\Delta\psi \sec^2 \phi d\phi - k^2 \tan \phi \sin \psi \cos \psi \frac{d\psi}{\Delta\psi} \right) + (---) \\ &= \frac{1}{1 + \tan^2 \phi \Delta^2 \psi} d(\tan \phi \Delta\psi) + (---) \\ &= d(\tan^{-1} \tan \phi \Delta\psi) + d(\tan^{-1} \tan \psi \Delta\phi) \\ \therefore \tan^{-1}(\tan \phi \Delta\psi) + \tan^{-1}(\tan \psi \Delta\phi) &= \mu; \end{aligned}$$

$$i. e. \quad \frac{\tan \phi \Delta\psi + \tan \psi \Delta\phi}{1 - \tan \phi \tan \psi \Delta\phi \Delta\psi} = \tan \mu.$$

And since evidently $\mu = \phi$ when $\psi = 0$, μ is the amplitude of $(u + v)$, where u, v are the elliptic functions whose amplitudes are ϕ, ψ respectively.

* (---) being the same function of (ψ, ϕ) that the preceding term is of (ϕ, ψ) .

The formula for $\text{sn}(u + v)$ can be very readily derived from above; thus,

$$\begin{aligned}
 \text{sn}(u + v) &= \sin \mu = \frac{\tan \mu}{\sqrt{1 + \tan^2 \mu}} \\
 &= \frac{\tan \phi \Delta \psi + \tan \psi \Delta \phi}{\sqrt{(\tan \phi \Delta \psi + \tan \psi \Delta \phi)^2 + (1 - \tan \phi \tan \psi \Delta \phi \Delta \psi)^2}} \\
 &= \frac{\tan \phi \Delta \psi + \tan \psi \Delta \phi}{\sqrt{1 + \tan^2 \phi \Delta^2 \psi + \tan^2 \psi \Delta^2 \phi + \tan^2 \phi \tan^2 \psi \Delta^2 \phi \Delta^2 \psi}} \\
 &= \frac{\tan \phi \Delta \psi + \tan \psi \Delta \phi}{\sqrt{(1 + \tan^2 \phi \Delta^2 \psi)(1 + \tan^2 \psi \Delta^2 \phi)}} \\
 &= \frac{\sin \phi \cos \psi \Delta \psi + \cos \phi \sin \psi \Delta \phi}{\sqrt{(\cos^2 \phi + \sin^2 \phi \Delta^2 \psi)(\cos^2 \psi + \sin^2 \psi \Delta^2 \phi)}} \\
 &= \frac{\sin \phi \cos \psi \Delta \psi + \cos \phi \sin \psi \Delta \phi}{1 - k^2 \sin^2 \phi \sin^2 \psi},
 \end{aligned}$$

since $\cos^2 \phi + \sin^2 \phi \Delta^2 \psi = 1 - k^2 \sin^2 \phi \sin^2 \psi = \cos^2 \psi + \sin^2 \psi \Delta^2 \phi$;
i. e. $\text{sn}(u + v) = \text{sn } u \text{ cn } v \text{ dn } v + \text{sn } v \text{ cn } u \text{ dn } u \div 1 - k^2 \text{sn}^2 u \text{sn}^2 v$.

The $\text{cn}(u + v)$ and $\text{dn}(u + v)$ can of course be just as readily obtained.